Some cases of the Lonely Runner conjecture

Introduction

For n points starting at one point and going at constant speed around a circle, the conjecture asks whether there is always a time when each point is arc distance at least C/n from the others, with C the circumference. The conjecture is true in the special case when two speeds are identical. Using convergence and ordinary Galilean relativity (choice of coordinates) we may assume that any arbitrary runner is stationary and the others have a set S of speeds, where S is a set of strictly positive integers whose greatest common divisor is equal to 1. The reason they may be taken positive is that positions only matter at a discrete set of times when we check that the end-point of one time interval is included in the next, and speeds only matter modulo their least common multiple. If the conjecture can be established for runner zero for all such sets S the conjecture will be proven.

Theorem 6 below will prove the statement of the Lonely Runner conjecture (for our arbitrarily chosen runner zero) for every possible set of nonzero speeds $S \subset \{1, 2, 3, 4, 5, 6, ..., N\}$ (assuming as we may that the elements of S have highest common divisor 1 and that the largest element of S is N), such that S has more than $N/2$ elements but does not contain the subset $\{1, 2\}$.

The cases which we leave unexamined are when $\{1, 2\} \subset S$, when one can restrict to the union of two large intervals of time $(\frac{1}{n}, \frac{1}{2} - \frac{1}{2n}) \cup (\frac{1}{2} + \frac{1}{2n}, 1 - \frac{1}{n})$ when runners 1 and 2 are far from runner zero, and the case when S has fewer than $\frac{N+2}{2}$ elements, when small intervals of time when runner 0 is far from the others begin to proliferate.

Interpretation as area.

Let the elements of S in order of size be $b_1, ..., b_{n-1}$, so the full set of speeds including that of runner zero is $\{0, b_1, b_2, ..., b_{n-1}\}$. On a real (x, y) Cartesian plane where the horizontal axis describes speed and the vertical axis describes arc distance divided by n, draw vertical lines described by the equations $x = b_1, x = b_2, ..., x = b_{n-1}$ and imagine a line through the origin rotating counter-clockwise, so the slope of the line represents time. The set of speeds $\{0, b_1, ..., b_{n-1}\}$ describes a counterexample just if the images of the open real intervals from $(ny - 1)/b_i$ to $(ny + 1)/b_i$ for integer values of y cover the positive real timeline. This is true just if the top limit point of each interval (we can choose top or bottom here) is contained in another interval. That is, for all integers u and subscripts i there must be an integer v and subscript j so that $\frac{nv-1}{b_j} < \frac{nu+1}{b_i} < \frac{nv+1}{b_j}$, equivalently

$$0 < det \begin{pmatrix} nu+1 & b_j \\ nv+1 & b_i \end{pmatrix} < 2b_i.$$

The left side is $det \begin{pmatrix} 1 & b_j \\ 1 & b_i \end{pmatrix} + n \cdot det(\begin{pmatrix} u & b_i \\ v & b_j \end{pmatrix}$ so the condition is also equivalent to $b_j - b_i < n \cdot det \begin{pmatrix} u & b_i \\ v & b_j \end{pmatrix} < b_j + b_i$ As long as we choose $b_j > b_i$ and choose u, v to make the determinant positive, the equation says that

$$np < b_i + b_j \qquad (1)$$

Here we take for p the area, the same as the number of whole or part integer lattice points in the paralellogram spanned by (u, b_i) and (v, b_j), with a lattice point on an edge counting as half a lattice point, and a lattice point on a corner calculated in proportion to the subtended angle.

A fundamental domain.

The picture is symmetrical with respect to the elementary transformation $(x, y) \mapsto (x, y + x)$.

We will call a point (b_i, y) *primitive* if it is not a positive integer multiple of a smaller similar point. Let's label our primitive integer points with slope between 0 and 1 as $(B_1, J_1), ..., (B_N, J_N)$. Then $B_i \in \{b_1, b_2, ..., b_{n-1}\}$ and we have

$$
\begin{aligned}
(B_1, J_1) &= (b_1, 0) \\
(B_2, J_2) &= (b_{n-1}, 1) \\
&\;\;... \\
(B_N, J_N) &= (b_1, b_1)
\end{aligned}
$$

All except the last one, that is $(B_1, J_1), ..., (B_{N-1}, J_{n-1})$, comprise a list of all the primitive points of slope greater than or equal to zero, but less than 1. There is, incidentally, a toric fan with edges the rays where the first quadrant meets lines of slope zero and one through the origin.

Toric interpretation (optional).

We could apply a second elementary transformation $(x, y) \mapsto (x - y, y)$ and we would find that this fan is reminiscent of the fan which resolves a cyclic quotient singularity. The monomials here in the dual lattice which are involved are those $x^i y^j$ for $i+j \in \{b_1, ..., b_{n-1}\}$ and we have a proper birational map to the affine plane with exceptional rational curves corresponding to $B_1, ..., B_{N-1}$. The cones of the fan are indexed by the rightmost primitive element of each cone which is $B_1, B_2, ..., B_{N-1}$ and so we see that there are $N - 1$ cones. The area of the basic paralellogram $A_i = B_i J_{i+1} - B_{i+1} J_i$ for $i = 1, 2, ..., N - 1$ is also called the index of the cone, if we were to construct the algebraic variety it would be the index of a corresponding singular point.

The speeds $\{0, b_1, ..., b_{n-1}\}$ are a lonely runner counterexample if and only if

$$A_i < \frac{B_i + B_{i+1}}{n}$$

for $i = 1, 2, ..., N - 1$.

Introduction of ϵ.

Let $0, b_1, b_2, ..., b_{n-1}$ with no common divisor larger than 1 with the b_i whole numbers such that $0 < b_1 < b_2 ... < b_{n-1}$. For each i when we consider the consecutive points $(B_i, J_i), (B_{i+1}, J_{i+1})$ we have that J_{i+1} is the smallest integer strictly larger than $J_i \frac{B_{i+1}}{B_i}$. Define ϵ_i with $0 < \epsilon_i \leq 1$ to be the difference

$$\epsilon_i = J_{i+1} - J_i \frac{B_{i+1}}{B_i}.$$

1. Lemma. The formula holds $\epsilon_i = \frac{A_i}{B_i}$ and consequently Then the set of speeds $\{0, b_1, ..., b_{n-1}\}$ *fails* the Lonely Runner condition for runner zero if and only if for all i

$$\frac{B_{i+1}}{B_i} > n\epsilon_i - 1.$$

Proof. The first formula is the definition of A_i divided by B_i, the second is the rewriting of the condition $A_i < \frac{1}{n}(B_i + B_{i+1})$ using the first formula.

2. Remark. The product of the $\frac{B_{i+1}}{B_i}$ is equal to 1.

3. Remark. In the case $(0, b_1, b_2, ..., b_{n-1}) = (0, 1, 2, 3, 4, ..., n-1)$ all the inequalities hold except $\frac{B_2}{B_1} = n\epsilon_1 - 1$ holds as an equality with $B_2 = n-1$, $B_1 = 1$, $\epsilon_1 = 1$.

4. Corollary. Let $\{0, b_0, ..., b_{n-1}\}$ be whole numbers such that $0 < b_1 < ... < b_{n-1}$ and the b_i have no common divisor higher than 1. The statement of the Lonely Runner conjecture is true (for runner zero) for this set of speeds whenever $b_{n-1} \leq (n-1)b_1$.

Proof. This the condition of Lemma 1 when $i = 1$. We have $(B_1, J_1) = (b_1, 0)$ while $(B_2, J_2) = (b_{n-1}, 1)$. The value of ϵ_1 is $1 - 0 = 1$ so failure of the condition implies $b_{n-1} > (n-1)b_1$.

5. Corollary. Let $\{0, b_0, ..., b_{n-1}\}$ be whole numbers such that $0 < b_1 < ... < b_{n-1}$ and the b_i have no common divisor higher than 1. Suppose not all b_i are odd and let s be minimum such that b_s is even. The statement of the lonely runner conjectre is true (for runner zero) for this set of speeds whenever $b_{n-1} \leq (\frac{n}{2} - 1)b_s$.

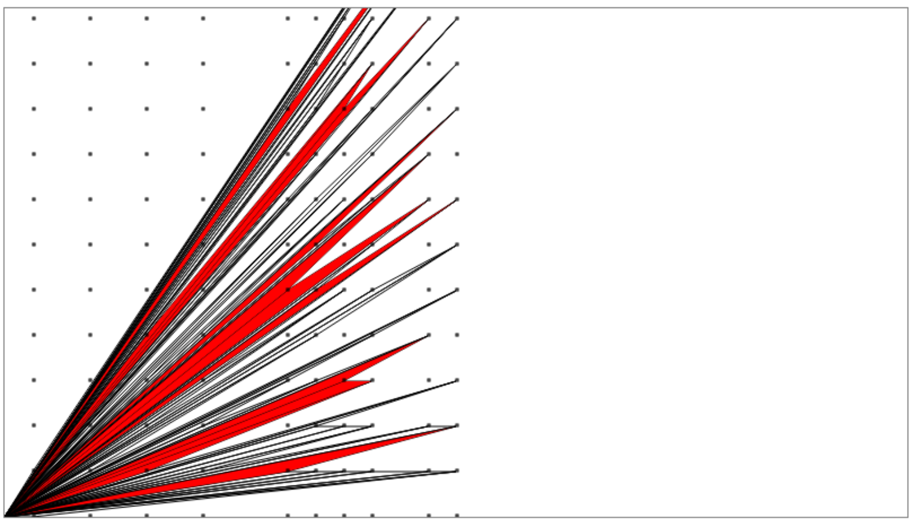
Proof. Since s is minimal the point $(b_s, \frac{b_s}{2})$ is suitably primitive that there is an i such that $(B_i, J_i) = (b_s, \frac{b_s}{2})$. Consider the consecutive points (B_i, J_i) and (B_{i+1}, J_{i+1}). Let t be such that $B_{i+1} = b_t$. Then J_{i+1} is the smallest integer strictly greater than $J_i \frac{b_t}{b_s}$ which evaluates to $\frac{b_t}{2}$. Thus if b_t is odd we have $\epsilon_i = \frac{1}{2}$ and if b_t is even we have $\epsilon_i = 1$. When the lonely runner condition for runner zero fails, Lemma 1 implies $\frac{B_{i+1}}{B_i} = \frac{b_t}{b_s} > n\epsilon - 1$. with the right side being either $n-1$ or $\frac{n}{2} - 1$. Then in any case $\frac{b_{n-1}}{b_s} \geq \frac{b_t}{b_s} > \frac{n}{2} - 1$.

6. Theorem. Let $S \subset \{1, 2, 3, 4, 5, 6, ..., N\}$, assuming as we may that $C = 1$, that the highest common divisor of the elements of S is 1 and the largest element of S is N. Suppose that S has more than $N/2$ elements but does not contain the subset $\{1, 2\}$. Then the statement of the Lonely Runner conjecture is true for the stationary runner when the set of nonzero speeds is S.

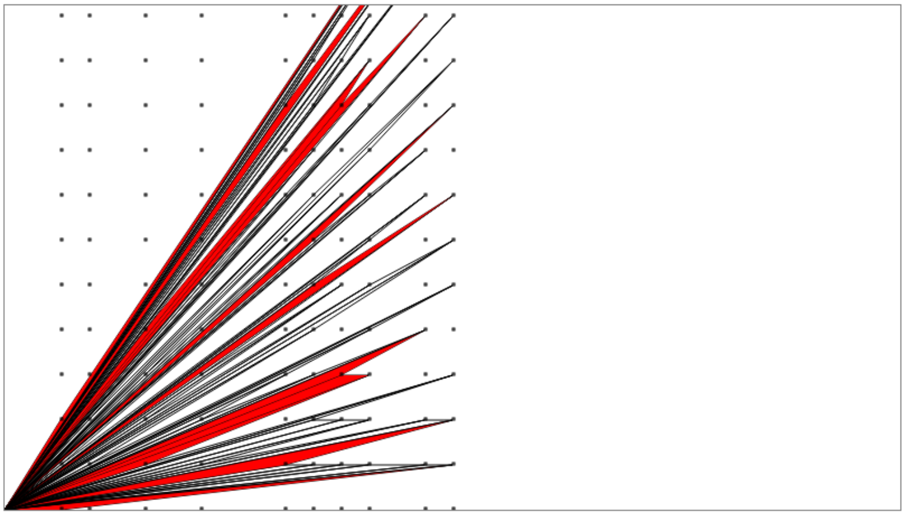
Proof. We have the set of speeds $\{0, b_1, ..., b_{n-1}\}$ such that $0 < b_1 < ... < b_{n-1}$ with all b_i whole numbers with no common divisor larger than 1. We have $b_{n-1} = N$ and the order of S is $n - 1 \geq N/2 + 1$. Then $2(n - 1) \geq b_{n-1} + 2$ so $b_{n-1} \leq 2n - 4$. If $b_1 \neq 1$ then $b_1 \geq 2$ and Corollary 4 proves the statement for all $b_{n-1} \leq (n - 1)b_1$ which is true if $b_{n-1} \leq (n - 1)2 = 2n - 2$ which certainly holds as we have $b_{n-1} \leq 2n - 4$. Now assume on the other hand that $b_1 = 1$. We are in case $b_2 \neq 2$. It is not possible that all b_i are odd, because that would require $b_{n-1} \geq 2n - 1$. Some of the b_i must be even. Let s be minimum so that b_s is even. Then $b_s \geq 4$. Corollary 5 proves the Lonely Runner statement for runner zero as long as $b_{n-1} \leq (\frac{n}{2} - 1)b_s$. Then it is proven for all $b_{n-1} \leq (\frac{n}{2} - 1) \cdot 4 = 2n - 4$ as required.

Examples.

Let's consider the ten-element subset $S = \{1, 3, 5, 7, 10, 11, 12, 13, 15, 16\} \subset \{1, 2, ..., 16\}$ The number of elements is greater than or equal to $\frac{16+2}{2}$ (it is greater, which is allowed), and S does not contain $\{1, 2\}$, so the hypothesis is satisfied and the lonely runner statement is true of runner zero. In this diagram slopes represent time, and lines through the origin while passing through triangles of large area – painted red– always hit a smaller interval of slope where runner zero is lonely. Since 10 is the smallest even element of S the point $(10, 5)$ is a vertex of a red triangle



Now we try instead $S = \{2, 3, 5, 7, 10, 11, 12, 13, 15, 16\} \subset \{1, 2, ..., 16\}$ This time 2 is included, but it is the smallest element of S and now $(1, 0)$ is a vertex of a red triangle.



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